Dynamics of nonlinear oscillators with random interactions

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(Received 3 October 1997)

We develop a mean field theory for a system of coupled oscillators with random interactions with variable symmetry. Numerical simulations of the resulting one-dimensional dynamics are in accordance with simulations of the N-oscillator dynamics. We find a transition in dependence on interaction strength J and symmetry parameter η from a dynamically disordered phase to a phase with static disorder, where all oscillators are frozen in random positions. This transition between the ''paramagnetic'' phase and the spin glass phase appears to be of first order and is dynamically characterized by chaos (positive Lyapunov exponents) in the former case and regular motion (vanishing Lyapunov exponents) in the latter case. The Lyapunov spectrum shows an interesting symmetry for antisymmetric interaction ($\eta=-1$). [S1063-651X(98)14608-1]

PACS number(s): 05.45.+b, 05.70.Fh, 64.60.Ht, 64.60.Cn

I. INTRODUCTION

Oscillations and interacting oscillating systems are omnipresent in nature as well as in technical systems. Therefore, systems of coupled oscillators have received much interest in the last years. Synchronization and desynchronization were investigated for populations of fireflies [1,2], pacemaker cells of the heart, and pulsating lasers [3,39]. Oscillations in the nervous system [5], which control periodical processes as running, breathing, and chewing, received particular interest. Recently it was conjectured that synchronization of oscillations plays a fundamental role in the mammalian brain. The binding of related features and the separation of unrelated features (binding-problem) could be achieved by synchronization and desynchronization of oscillating groups of neurons [6–8].

In the presence of dissipation, stable oscillations can only be generated by active systems, which have a limit cycle as attractor. Based on the idea of a phase description [9], ''phase models'' of coupled oscillators have been developed. Kuramoto [10] showed that any system of coupled limit cycle oscillators can be described in the limit of weak interaction by a set of first order differential equations of the oscillator phases ϕ_j. These models have been investigated mostly for uniform all-to-all interactions [10–17,35,42] but also for other connectivities [19,26]. To address the problem of interactions of oscillations in the brain, we consider the most natural choice of interactions, if apart from high connectivity no specific information about the interaction strengths is present: Gaussian random interactions with variable symmetry described by a symmetry parameter $\eta\in[-1,1]$. For symmetric interactions this kind of system was introduced by Daido [18], who found, for a sufficiently large average interaction strength J, a decay of ''magnetization'' $m=(1/N)\Sigma_{j=1}^{N}\exp(i\phi_j)$ according to a power law in time. Our simulations with identical system size, smaller time discretization, and a numerical procedure of higher order do not confirm this result. We find a power law only for a critical interaction strength J_c; for $J>J_c$ and $J<J_c$, we do find systematic deviations from a power law. At J_c, however, the system shows a discontinuous transition from a dynamically disordered state to a spin glass state with frozen disorder, in contrast to the case of uniform and Van Hemmen-type [20] interactions where the transition is continuous. In a wider perspective the considered model belongs to the large class of systems characterized by the interplay between frozen disorder and chaos [25,30,31].

The paper is organized as follows: In Sec. II, we describe the model in detail. A one-dimensional dynamics, which describes the interacting oscillators in the thermodynamic limit $N\rightarrow\infty$ exactly, is derived with the method of generating functionals [21,4] in Sec. III. Following the approach of Eissfeller and Opper [22] we performed numerical simulations of the one-dimensional dynamics (i.e., with $N=\infty$) for asymmetric interactions (η=0), which we compare with simulations of the N-oscillator dynamics (Sec. IV). In Sec. V, we show that the above mentioned transition is characterized by the dynamical EA order parameters as for XY spin glasses [23]. The different phases exist also for other symmetry parameters $\eta\neq1$. The part of parameter space (η,J), which shows spin-glass-like behavior corresponds dynamically to regular motion, while it is chaotic (maximal Lyapunov exponent $\lambda_{\max}>0$) in the ''paramagnetic'' case (Sec. VI). The Lyapunov spectrum shows an interesting symmetry for antisymmetric interactions ($\eta=-1$). Its origin is different from the one found in recently investigated systems [24]. In Sec. VII, we summarize our results.

II. MODEL

Kuramoto [10] showed that any system of coupled limit cycle oscillators can be described, in the limit of weak interaction, by a set of differential equations:

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$$\dot{\phi}_j(t)=\omega_j+F_j(\phi_1(t),\ldots,\phi_N(t)), \tag{1}$$

where $\phi_j(t)\in[0,2\pi)$ is the phase of the jth oscillator and ω_j its natural frequency. The coupling function F_j is 2π periodic in all arguments. Assuming that the interaction can be written as pair interaction, F_j can be expanded into a Fourier series [10]. Keeping only the first terms, one obtains the simple model equation

$$\dot{\phi}_j(t)=\omega_j+\sum_{i=1}^{N} J_{ij}\sin(\phi_i(t)-\phi_j(t)). \tag{2}$$

This model has mostly been investigated for uniform all-to-all interactions $J_{ij}=K/N$ [11–17]. In this case, for small interaction strength K, all oscillators are completely incoherent, the order parameter $m=\langle\exp(i\phi)\rangle:=(1/N)\Sigma_{j=1}^{N}\exp(i\phi_j)$ (denoted Z in the papers mentioned above) vanishes. Above a critical interaction strength K_c the system becomes partly coherent ($|m|>0$). At the critical interaction strength K_c the order parameter behaves as $|m|\sim\sqrt{(K-K_c)/K_c}$. It has recently been shown that this behavior occurs only for odd interaction functions (i.e., truncation after the first term of the Fourier expansion of F_j) [27–29]. For non-odd interaction function $|m|$ scales as $|m|\sim(K-K_c)/K_c$. Also, the case of Van Hemmen-type interactions $J_{ij}=K/N+C/N(\xi_i\eta_j+\xi_j\eta_i)$, with ξ_i and η_i independent identically distributed random variables that take values $+1$ and -1 with probability $\frac{1}{2}$, has been investigated [20]. Depending on the interaction strengths K and C, the system is in an incoherent, a partly coherent, a spin-glass-like, or a mixed state. The appropriate order parameters $q_1:=\langle\xi_j e^{i\phi_j}\rangle$ and $q_2:=\langle\xi_j e^{i\phi_j}\rangle$, which measure correlation with the interaction disorder, show up to a constant factor 2 the same dependence on the interaction strength C as $|m|$ does on K.

In the following we will analyze the case of Gaussian random interaction strengths J_{ij} and random frequencies ω_i with

$$[J_{ij}]=0,$$

$$[J_{ij}J_{kl}]=(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\eta)J^2/N,$$

$$[\omega_i]=\omega_0,$$

$$[\omega_i\omega_j]=\mu^2\delta_{ij}+\omega_0^2.$$

$[\cdot]$ denotes the quenched average over random frequencies and interaction strengths. For $\eta=1$ the interaction is symmetric, for $\eta=-1$ it is antisymmetric, and for $\eta=0$ the random variables J_{ij} and J_{ji} are uncorrelated. Without loss of generality one can assume $\omega_0=0$, which corresponds to the introduction of a rotating frame $\phi_j\to\phi_j-\omega_0 t$.

Interaction of two oscillators

To obtain a first understanding of the dynamics, one can investigate the behavior of two coupled oscillators. In this case Eq. (2) can be solved by introducing new variables $\Delta\phi=\phi_2-\phi_1$, $\overline{\phi}=(\phi_2+\phi_1)$, $\Delta J=J_{12}-J_{21}$, $\overline{J}=(J_{12}+J_{21})/2$, $\Delta\omega=\omega_2-\omega_1$, and $\overline{\omega}=(\omega_2+\omega_1)/2$. The equations of motion then read

$$\dot{\overline{\phi}}=\overline{\omega}+\Delta J\ \sin(\Delta\phi), \tag{3}$$

$$\Delta\dot{\phi}=\Delta\omega-2\overline{J}\sin(\Delta\phi). \tag{4}$$

The solution of Eq. (4) shows different behavior depending on the parameters. For sufficiently strong interaction, i.e., $|\Delta\omega/2\overline{J}<1|$, both oscillators move with a common frequency and constant phase difference:

$$\lim_{t\to\infty}\dot{\phi}_2-\dot{\phi}_1=\lim_{t\to\infty}\Delta\dot{\phi}=0. \tag{5}$$

This is called *phase locking*. A similar phenomenon also occurs for $N>2$ coupled oscillators. This is investigated for $N\to\infty$ in the next sections. For $|\Delta\omega/2\overline{J}|>1$ both oscillators move with different averaged frequencies:

$$\langle\dot{\phi}_2\rangle_t-\langle\dot{\phi}_1\rangle_t=\langle\Delta\dot{\phi}\rangle_t\neq 0. \tag{6}$$

This behavior is easily understood: If ϕ_1 and ϕ_2, and hence also $\Delta\phi$, are defined on $(-\infty,\infty)$, Eq. (4) can be regarded as gradient descent in a potential $V(\Delta\phi)$:

$$\dot{\Delta}\phi=-\frac{dV(\Delta\phi)}{d\Delta\phi}, \tag{7}$$

with

$$V(\Delta\phi)=-\Delta\omega\ \Delta\phi-2\overline{J}\cos(\Delta\phi). \tag{8}$$

The potential function $V(\Delta\phi)$ has the form of a tilted cosine function. For $|\Delta\omega|<2|\overline{J}|$ the potential function $V(\Delta\phi)$ has a local minimum (modulo 2π). As it is quadratic, the convergence to equilibrium is exponentially fast. For $|\Delta\omega|>2|\overline{J}|$, in contrast, $V(\Delta\phi)$ does not have local minima, and $\Delta\phi$ grows infinitely.

III. MEAN FIELD LIMIT

Due to the all-to-all interaction in Eq. (2), the dynamics is governed by a set of one-dimensional equations in the mean field limit $N\to\infty$. Following the usual approach of a dynamic mean field theory [21,22,30,31,4,32], first a Gaussian white noise $\xi_j(t)$ is introduced, which transforms Eq. (2) to a Langevin equation:

$$\dot{\phi}_j(t)=\omega_j+\sum_{i=1}^{N} J_{ij}\sin[\phi_i(t)-\phi_j(t)]+\xi_j(t), \tag{9}$$

$$\langle\xi_j(t)\xi_k(\hat{t})\rangle=\delta(t-\hat{t})\ \delta_{j,k}\ \sigma^2. \tag{10}$$

Averaged dynamical quantities can be obtained from the generating functional:

$$
\begin{aligned}
[Z]_{J,\omega}=\Bigg[\int D\phi(t)D\hat{\phi}(t)
&\times\exp\Bigg\{-\frac{\sigma^2}{2}\sum_j \int dt\ \hat{\phi}_j(t)^2\\
&-\sum_j \int dt\ i\hat{\phi}_j(t)\dot{\phi}_j(t)\\
&+\sum_j \int dt\ i\hat{\phi}_j(t)\omega_j\\
&+\sum_{j,i} J_{i,j}\int dt\ i\hat{\phi}_j(t)\\
&\times\sin[\phi_i(t)-\phi_j(t)]\Bigg\}\Bigg]_{J,\omega},
\end{aligned}
\tag{11}
$$

where $\int D\phi(t)=\lim_{\tau\to 0}\Pi_{i=1}^{N}\Pi_{t_j}\int(2\pi)^{-1/2}d\phi(t_j)$ denotes functional integration over all phase variables. Since J_{ij} and ω_j are Gaussian random variables, the average in Eq. (11) can be calculated (see the Appendix). The averaged generating functional $[Z]_{J,\omega}$ factorizes. Hence the dynamics of oscillators at different sites is independent. Therefore we can omit the site index i, and write

$$
[Z]_{J,\omega}=([Z]^1_{J,\omega})^N=\left(\int \Pi_s \frac{d\phi(t_s)d\hat{\phi}(t_s)}{2\pi}e^{-S_1[\phi,\hat{\phi},K,R]}\right)^N.
\tag{12}
$$

The one-dimensional averaged generating functional $[Z]^1_{J,\omega}$ corresponds to a dynamics, which obeys the following (generalized) Langevin equation, as can be seen by calculating the dynamical generating functional of a generalized Langevin equation with multidimensional Gaussian noise (see e.g., Refs. [32] and [33]):

$$
\begin{aligned}
\dot{\phi}(t)=J\ \mathrm{Re}(e^{i\phi(t)}\zeta(t))+\sigma\xi_3(t)+\mu\xi_4(t)\\
+\eta\frac{J^2}{2}\int_0^t d\tilde{t}\ \mathrm{Re}(e^{i\phi(t)}R_1(t,\tilde{t})e^{i\phi(\tilde{t})}\\
+e^{i\phi(t)}R_2(t,\tilde{t})e^{-i\phi(\tilde{t})}),
\end{aligned}
\tag{13}
$$

with the Gaussian noise variables $\xi_3,\xi_4\in\mathrm{R}$ and $\zeta\in\mathrm{C}$ with zero mean and the correlations

$$
\begin{aligned}
\langle\zeta(t)\zeta(\tilde{t})\rangle&=K_+(t,\tilde{t}):=\langle e^{i\phi(t)}e^{i\phi(\tilde{t})}\rangle,\\
\langle\zeta^*(t)\zeta(\tilde{t})\rangle&=K_-(t,\tilde{t}):=\langle e^{-i\phi(t)}e^{i\phi(\tilde{t})}\rangle,\\
\langle\xi_3(t)\xi_3(\tilde{t})\rangle&=\delta_{t,\tilde{t}},\\
\langle\xi_4(t)\xi_4(\tilde{t})\rangle&=1,\\
\langle\xi_3(t)\xi_4(\tilde{t})\rangle&=\langle\xi_3(t)n(\tilde{t})\rangle=\langle\xi_4(t)n(\tilde{t})\rangle=0,
\end{aligned}
\tag{14}
$$

and the complex response functions

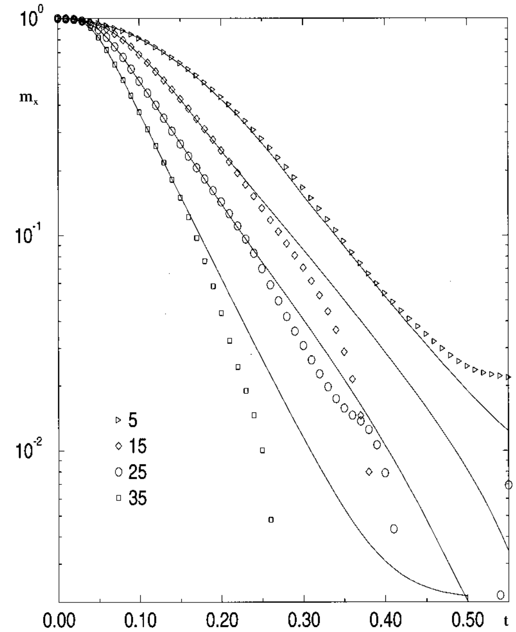

FIG. 1. Comparison of the magnetization $m_x(t)$ of the simulation of the N-particle system (symbols) and the effective one-particle system (lines) for $J=5$, 15, 25, and 35. For short times the agreement is very good; for larger times the fluctuations due to the finite size becomes apparent in the direct simulation.

$$
\begin{aligned}
R_1(t,\tilde{t})&=\langle i\hat{\phi}(\tilde{t})e^{-i\phi(t)}e^{-i\phi(\tilde{t})}\rangle,\\
R_2(t,\tilde{t})&=\langle i\hat{\phi}(\tilde{t})e^{-i\phi(t)}e^{i\phi(\tilde{t})}\rangle,
\end{aligned}
\tag{15}
$$

with $\langle\cdot\rangle$ denoting the average over the noise variables. The Gaussian noise $\xi_4(t)$ originates from the different undisturbed frequencies ω of the oscillators.

IV. NUMERICAL INTEGRATION OF THE ONE-DIMENSIONAL DYNAMICS

The one-dimensional dynamics (13), which describes the system of coupled oscillators in the thermodynamic limit $N\to\infty$, is integrated numerically following an approach developed in Ref. [22]. The procedure consists of simulating a large number M of one-dimensional trajectories in order to calculate the averages (14). Since the individual trajectories are statistically independent, the statistical error is expected to be of order $M^{-1/2}$ in contrast to a N-particle simulation of Eq. (9) which may show finite-size effects of unpredictable size [22].

For $\eta=0$ and $\sigma=0$ we investigate the decay of the magnetization $m_x=\langle\cos\phi\rangle$, $m_y=\langle\sin\phi\rangle$ and the correlation functions $K_{cc}(t,\hat{t})=\langle\cos\phi(t)\cos\phi(\hat{t})\rangle$, $K_{ss}(s,t)=\langle\sin\phi(t)\sin\phi(\hat{t})\rangle$, and $K_{sc}=\langle\sin\phi(t)\cos\phi(\hat{t})\rangle$ from the deterministic initial condition $\phi=0$ (i.e. $m=1$). The complex order parameter $m=m_x+im_y$, the magnetization for XY spins, is equivalent to the order parameter called Z by Kuramoto [10]. These initial conditions imply $m_x(t)=K_{cc}(0,t)$. Since in both cases the integrations are carried out numerically, the results correspond to very small noise rather than zero noise. Thus we cannot observe unstable dynamical behavior, which may be present for σ exactly zero. Correspondingly, simulations with very small $\sigma=10^{-12}\ll 1/M$ do

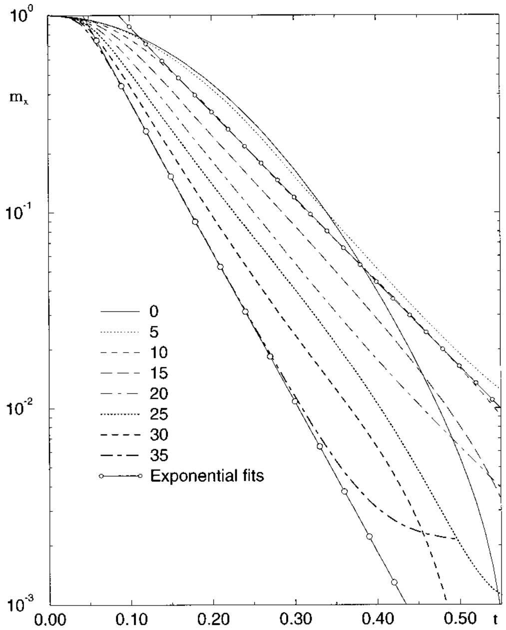

FIG. 2. Magnetization $m_x(t)$ depending on t for different interaction strengths J and for $\eta=0$, obtained from the effective one-particle system and fitted exponential functions. The coincidence with the exponential fits is very good for about two orders of magnitude.

not show significant deviations from simulations with $\sigma=0$. As an example, in Fig. 1 the results for $M=10^5$ and time discretization $\tau=0.005$ are compared to results obtained by a direct simulation of Eq. (9) with $N=500$ oscillators. For short times we find very good agreement; for larger times the fluctuations due to the finite size becomes apparent in the direct simulation.

For short times the frequency term is dominant since the interaction terms vanish for $m=1$. For $J=0$ $m(t)$ is the Fourier transform of the frequency distribution $g(\omega)$, and hence a Gaussian. For $J>0$ the decay of m obeys an exponential law after the short intermediate period, as is shown in Fig. 2 for different interaction strengths J. The coincidence with the exponential fits is very good for about two orders of magnitude. We measured the exponent a of the exponential decay for $J=5-50$. For large interaction strength, J shows the expected dependence $a\sim J$. In the case of $J\rightarrow\infty$ the different frequencies are negligible, and a transformation of $J\rightarrow sJ$ corresponds to a time scaling $t\rightarrow t/s$, hence the exponent a is proportional to J. The slope of the extrapolating line is $a_\infty=0.478\pm0.02$. In the case of $J\rightarrow\infty$ the magnetization decays as $m(t)\sim e^{-0.478\,t\,J}$.

V. PHASE LOCKING FOR SYMMETRIC INTERACTION

In the case $\eta=1$, we can identify a constant of motion:

$$\begin{aligned}\sum_i \dot{\phi}_i &= \sum_i \omega_i + \sum_{i,j} J_{ij}\sin(\phi_j-\phi_i)\\ &= \sum_i \omega_i + \sum_{i,j<i} \underbrace{(J_{ij}-J_{ji})}_{=0}\sin(\phi_j-\phi_i)\\ &= \sum_i \omega_i =: N\bar{\omega}. \end{aligned} \tag{16}$$

The average angular velocity and hence the average angle $\overline{\phi}:=(1/N)\Sigma_j\phi_j$ is independent of the dynamics of the other degrees of freedom. $\overline{\phi}-\overline{\omega}t$ is a *constant of motion*.

This constant of motion suggests the introduction of quantities in a rotating frame, i.e., $\phi\rightarrow\tilde{\phi}=\phi-\overline{\omega}t$. A freezing of oscillators in random position can be described by the order parameters

$$\begin{aligned}\tilde{q}^x &= \lim_{t\to\infty}\lim_{N\to\infty}\langle\cos(\tilde{\phi}_i(t_0))\cos(\tilde{\phi}_i(t_0+t))\rangle\\ &= \lim_{t\to\infty}\lim_{N\to\infty}\langle\cos(\phi_i(t_0)-\overline{\omega}t_0)\cos(\phi_i(t_0+t)-\overline{\omega}(t_0+t))\rangle,\end{aligned} \tag{17}$$

$$\begin{aligned}\tilde{q}^y &= \lim_{t\to\infty}\lim_{N\to\infty}\langle\sin(\tilde{\phi}_i(t_0))\sin(\tilde{\phi}_i(t_0+t))\rangle\\ &= \lim_{t\to\infty}\lim_{N\to\infty}\langle\sin(\phi_i(t_0)-\overline{\omega}t_0)\sin(\phi_i(t_0+t)-\overline{\omega}(t_0+t))\rangle\end{aligned} \tag{18}$$

$$\tilde{m}_x=\langle\cos(\tilde{\phi}_i(t))\rangle=\langle\cos(\phi_i(t)-\overline{\omega}t)\rangle, \tag{19}$$

$$\tilde{m}_y=\langle\sin(\tilde{\phi}_i(t))\rangle=\langle\sin(\phi_i(t)-\overline{\omega}t)\rangle. \tag{20}$$

Here $\langle\cdot\rangle$ denotes the average over all oscillators. The order parameters $\tilde{q}_x$ and $\tilde{q}_y$ are equivalent to the dynamical Edwards Anderson order parameters for XY spins (see, e.g., Ref. [37]).

The order parameter

$$\tilde{q}:=\tilde{q}^x+\tilde{q}^y, \tag{21}$$

is invariant under rotation of all oscillators. $\tilde{q}=0$ implies the absence of spin glass order. This quantity can be calculated directly from the correlation function

$$\tilde{K}_-(t,\hat{t})=\langle e^{-i(\phi(t)-\overline{\omega}t)}e^{i(\phi(\hat{t})-\overline{\omega}\hat{t})}\rangle \tag{22}$$

as

$$\tilde{q}=\lim_{t\to\infty}\lim_{N\to\infty}\mathrm{Re}\,\tilde{K}_-(t_0,t_0+t). \tag{23}$$

The order parameter $\tilde{q}\in[0,1]$ vanishes, if all correlations between different oscillators decay, it equals 1 if the phase differences between different oscillators are constant. The absolute value of the order parameter $\tilde{m}=\tilde{m}_x+i\tilde{m}_y$ vanishes if there is no remanent magnetization, i.e., all phases are equally frequent. It is 1 if the phases of all oscillators are equal. As the calculation of the response functions is numerically very expensive and phase locking only occurs for very large times inaccessible to a numerical integration of the one-dimensional dynamics, we directly simulate Eq. (9) with the Heun method (see, e.g., Ref. [34]).

We calculate $\tilde{K}_-(T,t+T)$ for $N=100$, 200, and 400 oscillators over a time of $t=1000$ after the system has relaxed. Figure 3 shows $\tilde{q}$ depending on the interaction strength J for $N=100$, 200, and 400. The transition from

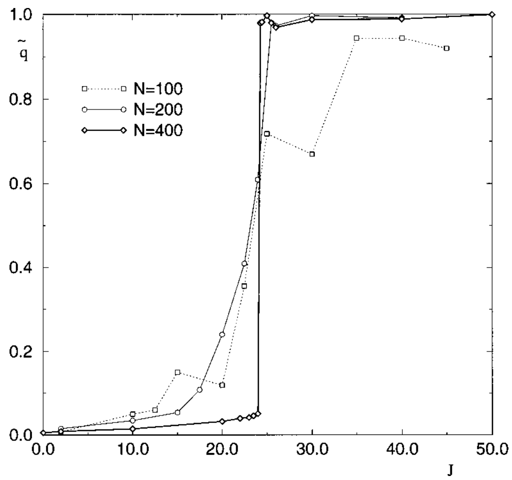

FIG. 3. Order parameter $\tilde{q}$ depending on the interaction strength J for $N=100$, 200, and 400 oscillators. The transition from $\tilde{q}\approx 0$ for $J<J_c$ to $\tilde{q}\approx 1$ for $J>J_c$ becomes sharper with growing system size N. Between $J=24$ and 24.3, the order parameter $\tilde{q}$ changes from 0.046 to 0.982.

$\tilde{q}\approx 0$ for $J<J_c$ to $\tilde{q}\approx 1$ for $J>J_c$ becomes sharper with growing system size N. Between $J=24$ and 24.3, $\tilde{q}$ changes from 0.046 to 0.982. This calculation was carried out with 15 different interaction matrices; the results are the same as in Fig. 3, but the critical interaction strength varied slightly.

A. Decay of magnetization

The numerical calculation shows no remanent magnetization m. There is no significant difference between simulations started with initial condition $m(0)=1$ and random initial phases $\phi_i(0)$ [magnetization $m(0)=0$].

We investigate the decay of $m_x(t)=\langle \cos\phi(t)\rangle$ for an initial condition $m_x(0)=1$. For this case, Daido [18] found an exponential decay of the magnetization for $0.7<m_x<0.07$ and for interaction strength $J<J_c=6.5$ (for $N=1000$), while for $J>J_c$ it obeyed a power law.

Our simulations, with identical system size, smaller time discretization (10^{-3} vs 10^{-2}), and a numerical procedure of higher order, do not confirm this result. An evaluation of integration procedures of different order and different time steps yields that the Euler integration scheme with $\Delta t=10^{-2}$ as used by Daido shows strong discretization effects (see Fig. 4). Our results are shown in Fig. 5. Only for $J=J_c$ does the magnetization m_x obey a power law $m_x(t)\sim t^{\alpha}$. The exponent is determined from a fit in the log-log plot as $\alpha=1.01\pm 0.02$. While for $J<J_c$ we can confirm the exponential decay of the magnetization as found in Ref. [18], the behavior for $J>J_c$ appears to be more complex. In this region we find strong fluctuations around a pure power law. Although this result is not in contradiction to the existence of a glassy phase, a definite answer to the question of the asymptotic behavior of the magnetization would require simulations with much larger system sizes.

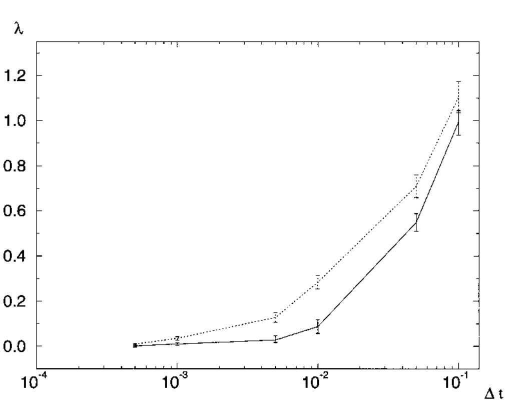

FIG. 4. Dependence of the largest Lyapunov exponent λ on the time step Δt for Euler (dashed curve) and Heun integration schemes. The Euler integration scheme with $\Delta t=10^{-2}$ as used by Daido shows strong discretization effects.

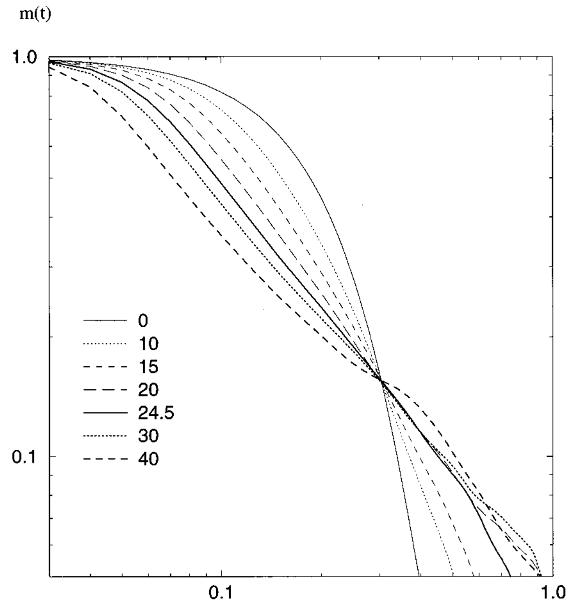

FIG. 5. Doubly logarithmic plot of the magnetization m_x for $J=0$, 10, 15, 24, 30, and 40. Only for $J=J_c$ does the magnetization m_x obey a power law $m_x(t)\sim t^{\alpha}$.

B. Almost symmetric interaction matrix

Numerical investigations for almost symmetric interaction matrices show that phase locking can also occur for $\eta<0$ above a critical interaction strength $J_c(\eta)$. In numerical experiments, one has to work with a finite time discretization τ. We observe an intermittency phenomenon, which does not occur for $\eta=0$ and strongly depends on τ. After a time t_{regular} the regular motion breaks down, but after a short time it recovers again (see Fig. 6). Note that $|K_-(T,T+t)|$ grows to 1 again, hence the oscillators approach *the same configura-*

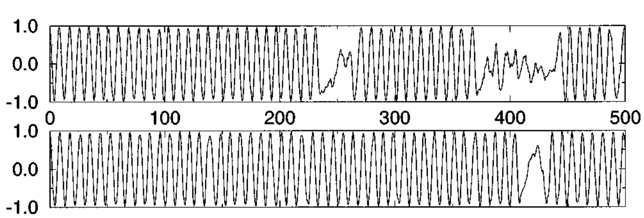

FIG. 6. Re $K_-(T,T+t)$ for $T=1000$ and $\Delta t=0.01$ (top) or 0.005 (bottom) and $J=30$ and $\eta=0.9$. The intermittency phenomenon is strongly dependent on the time discretization Δt.

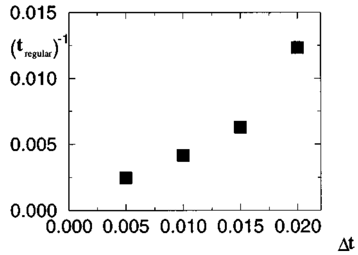

FIG. 7. The dependence of the duration of regular motion t_{regular} on the time discretization Δt suggests that t_{regular} diverges to infinity for $\Delta t \to 0$. Plotted is $(t_{\text{regular}})^{-1}$ against Δt.

tion as before the breakdown. This gives rise to the speculation, that the system for $\eta = 1$ finds a minimum of the energy landscape.

The strong dependence of the intervals between breakdowns t_{regular} of the regular motion on τ is shown in Fig. 7. We conjecture that t_{regular} diverges for $\tau \to 0$.

The critical interaction strength $J_c(\eta)$ is strongly dependent on η. With the growth of the asymmetric portion of the interaction matrix, $J_c(\eta)$ becomes larger. For $\eta < 0.8$, freezing of oscillators does not occur. This result corresponds to numerical results of Kinzel and Spitzner [36], who found spin glass order in the SK model with Ising spins for $\eta > 0.83 \pm 0.02$.

Our results are summarized in Fig. 8. Above $J_c(\eta)$ the oscillators are frozen in random positions like spins in a spin glass. This part of phase space is denoted by SG. The order parameter $\tilde{q}$ is 1, and the magnetization m vanishes. For $J \leq J_c(\eta)$ (denoted ''paramagnetic'' in Fig. 8), both $\tilde{q}$ and m vanish. The motion of the oscillators is completely incoherent.

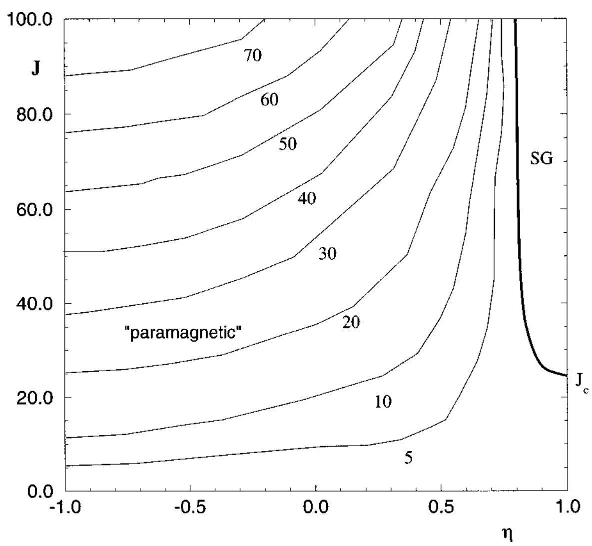

FIG. 8. Value of the largest Lyapunov exponent $\lambda_{\max}$ depending on the symmetry parameter η and interaction strength J represented by contour lines at levels $\lambda_{\max} = 5$, 10, 20, 30, 40, 50, 60, and 70 ($N = 100$). This graph is obtained for one fixed disorder realization of the frequencies and couplings by varying J and η [see Eq. (A7)]. In order to suppress the (rather small) sample to sample fluctuations, we averaged over three disorder realizations. The part of the parameter space (η, J) above $J_c(\eta)$ (bold line), where the oscillators freeze in random positions, is denoted by ''SG.'' In this regime $\lambda_{\max} = 0$. For $\eta < 0.8$ there is no spin glass order. The largest Lyapunov exponent decreases monotonically with η and, for $\eta < 0.8$ grows monotonically with J.

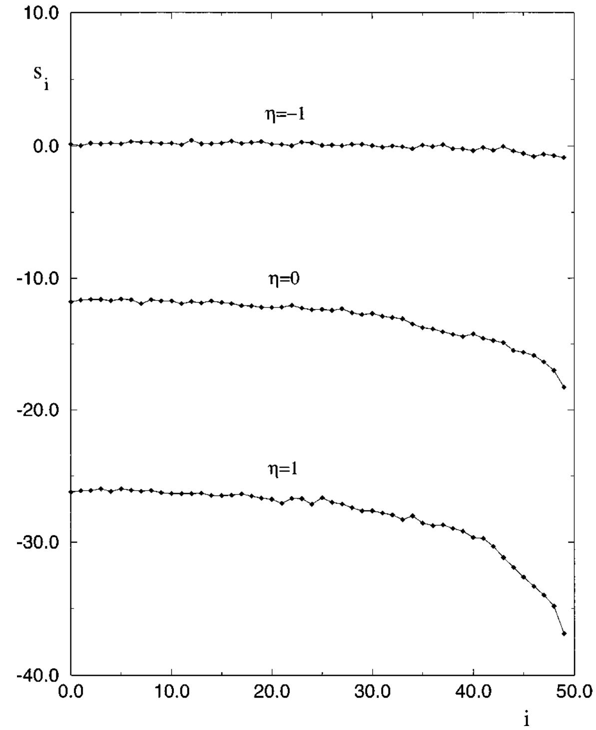

FIG. 9. Symmetry values $s_i = \lambda_{N/2+i} - \lambda_{N/2-i-1}$ for $i = 0, \ldots, 49$ calculated from the ordered spectrum of Lyapunov exponents λ_i of $N = 100$ oscillators ($J = 10$). Only for $\eta = -1$ is the Lyapunov spectrum symmetric.

VI. DYNAMIC PROPERTIES

To investigate the dynamic properties, the Lyapunov spectra are calculated for different interaction strengths J and symmetry parameters η. These numerical results suggest that the Lyapunov exponents are dense in the limit $N \to \infty$. There is no hint of a discrete component, as was reported by for a system of coupled oscillators with phase and amplitude variables [37].

Our numerical investigations show that for $\eta = -1$ the Lyapunov spectrum is symmetric. This is not only valid in the thermodynamic limit $N \to \infty$, but also for small N, e.g., $N = 4$. The symmetry can be quantified by symmetry values $s_i := \lambda_{N/2+i} - \lambda_{N/2-i-1}$, $(i = 0, \ldots, N/2 - 1$ Neven) calculated from the ordered spectrum of Lyapunov exponents λ_i. Only for antisymmetric interaction matrix ($\eta = -1$) do we find $s_i = 0$ (up to numerical inaccuracies). The symmetry also holds for odd N, but with one unpaired zero Lyapunov exponent. For symmetry parameter $\eta \neq -1$ we do find systematic deviations from $s_i = \text{const}$. This is shown for three examples in Fig. 9.

One can easily understand that the sum of all Lyapunov exponents must vanish for $\eta = -1$, since the local contraction rate of the phase space volume, the divergence, vanishes:

$$\text{div}F(\phi) = \sum_{j=1}^{N} \frac{\partial F_j}{\partial \phi_j} = \sum_{j=1}^{N} \sum_{l=1}^{N} J_{lj} \cos(\phi_j - \phi_l)$$
$$= \sum_{j=1}^{N} \sum_{l=1}^{j-1} (J_{lj} + J_{jl}) \cos(\phi_j - \phi_l) = 0. \quad (24)$$

In contrast to the conservation of volume in phase space, we find the symmetry of the spectrum is no short time property. Differing from Hamiltonian systems, where the symmetry is a short time property, the Jacobi matrix does not have a symplectic structure. In recent publications other systems were described which show symmetric Lyapunov spectra [38,39,40,41]. A criterion developed in Ref. [24] for the occurrence of a symmetric Lyapunov spectrum, which is a generalization of the infinitesimal symplectic condition for the Jacobian, is not fulfilled here, since the short time approximation of the Lyapunov spectrum is not symmetric, which would be required.

VII. CONCLUSION

We developed a mean field theory for a system of coupled oscillators with Gaussian random interactions with variable symmetry. With the method of generating functionals [21,4], we derived a one-dimensional dynamics which describes the interacting oscillators in the thermodynamic limit $N\rightarrow\infty$ exactly. The numerical simulations of the one-dimensional dynamics corresponding to $N\rightarrow\infty$ and for asymmetric interactions conform with simulations of the N-oscillator dynamics. The exponential decrease of the correlation functions can be verified for almost two orders of magnitude, as the fluctuations due to a finite number of simulated paths are much smaller than for a simulation of the N-oscillator dynamics (for the same amount of computer memory used).

For symmetric interaction we find a transition from an incoherent state to a state where all oscillators are frozen in random positions like spins in a spin glass depending on the interaction strength J. It is discontinuous, in contrast to the case of uniform interactions (Kuramoto model), where there is a continuous transition from an incoherent to a partly coherent state [10].

ACKNOWLEDGMENTS

We thank H. G. Schuster and L. Molgedey for helpful discussions.

APPENDIX: DERIVATION OF THE MEAN FIELD DYNAMICS

The effective one-dimensional dynamics can be deduced in the following way.

(i) Calculation of the generating functional Z.

(ii) Calculation of the quenched average of Z over random frequencies and random interactions.

(iii) Writing the exponent $[Z]_{\omega,J}$ in a form with squares of sums of *local* quantities.

(iv) Linearization of the exponent of $[Z]_{\omega,J}$ in these sums of local quantities with a functional Hubbard-Stratonovich transformation (HST).

(v) Coordinate transformation in $Q_\alpha(t)$.

(vi) Performing the integrations over $\hat{Q}_\alpha(t,\hat{t})$ with the method of steepest descent.

(vii) Gathering the one-dimensional dynamics from factorizing $[Z]_{\omega,J}$.

As the effective one-dimensional dynamics will be solved numerically, and the equation in the Ito interpretation can be integrated much more easily, we start with a discretized equation in the Ito interpretation. The generalization to the Stratonovich interpretation and the continuous time limit is straightforward. The discretized Langevin equation reads:

$$\phi_j(t_{s+1})=\phi_j(t_l)+\omega_j\tau + \tau\sum_i J_{i,j}\sin(\phi_i(t_s)-\phi_j(t_s))+\xi_j(t_{s+1}), \tag{A1}$$

with Gaussian noise $\langle\xi_j(t_s)\xi_j(t_{\hat{s}})\rangle=\delta_{s,\hat{s}}\ \delta_{i,j}\ \sigma^2\tau$. The generating functional Z is

$$Z=\int \Pi_s\Pi_k\frac{d\phi_k(t_s)d\hat{\phi}_k(t_s)}{2\pi}\exp\Bigg\{-\frac{\sigma^2}{2}\sum_{s,j}\ \tau\hat{\phi}_j^2(t_s)-\sum_{s,j}\ \tau i\hat{\phi}_j(t_s)\frac{\phi_j(t_s)-\phi_j(t_{s-1})}{\tau} + \sum_{s,j}\ \tau i\hat{\phi}_j(t_s)\omega_j+\sum_{s,j}\ \tau i\hat{\phi}_j(t_s)\sum_k\ J_{k,j}\sin(\phi_k(t_{s-1})-\phi_j(t_{s-1}))\Bigg\}. \tag{A2}$$

First Z is averaged over the random frequencies, which obey $p(\omega_i)=(1/\sqrt{2\pi}\mu)\,e^{-(\omega_i^2/2\mu^2)}$. We find

$$\begin{aligned}\left[\exp\left(\sum_j\ dt\ \hat{i}\phi_j(t)\omega_j\right)\right]_\omega &=\Pi_j\int d\omega_j p(\omega_j)\exp\left(\omega_j\sum_s\ \tau i\hat{\phi}_j(t)\right)\\ &=\Pi_j\int d\omega_j\frac{1}{\sqrt{2\pi}\mu}\exp\left(-\frac{\omega_i^2}{2\mu^2}+\omega_j\sum_s\ \tau i\hat{\phi}_j(t)\right)\\ &=\Pi_j\exp\left(\frac{\mu^2}{2}\left[\sum_s\ \tau i\hat{\phi}_j(t)\right]^2\right)\end{aligned} \tag{A3}$$

$$=\exp\Bigg\{-\frac{\mu^2}{2}\sum_j\ \sum_s\ \tau\sum_{\tilde{s}}\ \tau\ \hat{\phi}_j(t_s)\hat{\phi}_j(t_{\tilde{s}})\Bigg\}. \tag{A4}$$

A non-Gaussian distribution function $p(\omega_i)$ would lead to higher powers of $\hat{\phi}_j$ in the exponent:

$$[Z]_\omega = \int \Pi_s \Pi_k \frac{d\phi_k(t_s) d\hat{\phi}_k(t_s)}{2\pi} \exp\left\{ -\tilde{S}_0 + \sum_s \tau \sum_{k,j} i\hat{\phi}_j(t_s) J_{k,j} \sin(\phi_k(t_{s-1}) - \phi_j(t_{s-1})) \right\}, \tag{A5}$$

with the J_{kj}-independent term

$$\tilde{S}_0 = \sum_j \frac{\sigma^2}{2} \sum_s \tau \hat{\phi}_j^2(t_s) + \sum_s \tau i\hat{\phi}_j(t_s) \frac{\phi_j(t_s) - \phi_j(t_{s-1})}{\tau} + \frac{\mu^2}{2} \sum_{s,\tilde{s}} \tau^2 \hat{\phi}_j(t_s) \hat{\phi}_j(t_{\tilde{s}}). \tag{A6}$$

$\tilde{S}_0$ is a sum of one particle terms. The interaction matrix J_{ij} can be written as sum of a symmetric matrix and an antisymmetric matrix:

$$J_{ij} = \frac{\sqrt{1+\eta}}{\sqrt{2}} J_{ij}^{(s)} + \frac{\sqrt{1-\eta}}{\sqrt{2}} J_{ij}^{(\mathrm{as})}$$

with (A7)

$$J_{ij}^{(s)} = J_{ji}^{(s)} \quad \text{and} \quad J_{ij}^{(\mathrm{as})} = -J_{ji}^{(\mathrm{as})}.$$

Hence $[Z]_\omega$ can be written as

$$[Z]_\omega = \int \Pi_s \Pi_k \frac{d\phi_k(t_s) d\hat{\phi}_k(t_s)}{2\pi} \exp\left\{ -\tilde{S}_0 + \sum_s \tau \sum_{j,k<j} \frac{\sqrt{1+\eta}}{\sqrt{2}} J_{k,j}^{(s)} (i\hat{\phi}_j(t_s) - i\hat{\phi}_k(t_s)) \sin(\phi_k(t_{s-1}) - \phi_j(t_{s-1})) \right.$$
$$\left. + (i\hat{\phi}_j(t_s) + i\hat{\phi}_k(t_s)) \frac{\sqrt{1-\eta}}{\sqrt{2}} J_{k,j}^{(as)} \sin(\phi_k(t_{s-1}) - \phi_j(t_{s-1})) \right\}. \tag{A8}$$

The average over J_{kj} can now be carried out as $N(N-1)$ Gaussian integrations:

$$[Z]_{\omega,J} = \int \Pi_s \Pi_k \frac{d\phi_k(t_s) d\hat{\phi}_k(t_s)}{2\pi} \exp\left\{ -\tilde{S}_0 + \frac{J^2}{4N} \left(\sum_{j,k} \frac{1+\eta}{2} \sum_{s,\tilde{s}} \tau^2 (i\hat{\phi}_j(t_s) - i\hat{\phi}_k(t_s))(i\hat{\phi}_j(t_{\tilde{s}}) - i\hat{\phi}_k(t_{\tilde{s}})) \right.\right.$$
$$\times \sin(\phi_k(t_{s-1}) - \phi_j(t_{s-1})) \sin(\phi_k(t_{\tilde{s}-1}) - \phi_j(t_{\tilde{s}-1})) + \sum_{j,k} \frac{1-\eta}{2} \sum_{s,\tilde{s}} \tau^2 (i\hat{\phi}_j(t_s) + i\hat{\phi}_k(t_s))(i\hat{\phi}_j(t_{\tilde{s}}) + i\hat{\phi}_k(t_{\tilde{s}}))$$
$$\left.\left. \times \sin(\phi_k(t_{s-1}) - \phi_j(t_{s-1})) \sin(\phi_k(t_{\tilde{s}-1}) - \phi_j(t_{\tilde{s}-1})) \right) \right\} \tag{A9}$$

$$= \int \Pi_s \Pi_k \frac{d\phi_k(t_s) d\hat{\phi}_k(t_s)}{2\pi} \exp\left\{ -\tilde{S}_0 + \frac{J^2}{2N} \sum_{s,\tilde{s}} \tau^2 \sum_{j,k} (i\hat{\phi}_j(t_s) i\hat{\phi}_j(t_{\tilde{s}}) + i\hat{\phi}_k(t_s) i\hat{\phi}_k(t_{\tilde{s}}) \right.$$
$$\left. - \eta\, i\hat{\phi}_j(t_s) i\hat{\phi}_k(t_{\tilde{s}}) - \eta\, i\hat{\phi}_k(t_s) i\hat{\phi}_j(t_{\tilde{s}})) \sin(\phi_k(t_{s-1}) - \phi_j(t_{s-1})) \sin(\phi_k(t_{\tilde{s}-1}) - \phi_j(t_{\tilde{s}-1})) \right\}. \tag{A10}$$

With the definitions

$$\tilde{K}_{cc}(t_s, t_{\tilde{s}}) = \frac{1}{N} \sum_j \cos \phi_j(t_{s-1}) \cos \phi_j(t_{\tilde{s}-1}), \tag{A11}$$

$$\tilde{K}_{ss}(t_s, t_{\tilde{s}}) = \frac{1}{N} \sum_j \sin \phi_j(t_{s-1}) \sin \phi_j(t_{\tilde{s}-1}), \tag{A12}$$

$$\tilde{K}_{sc}(t_s, t_{\tilde{s}}) = \frac{1}{N} \sum_j \sin \phi_j(t_{s-1}) \cos \phi_j(t_{\tilde{s}-1}), \tag{A13}$$

$$\tilde{R}_{cc}(t_s, t_{\tilde{s}}) = \frac{1}{N} \sum_j i\hat{\phi}_j(t_{\tilde{s}}) \cos \phi_j(t_{s-1}) \cos \phi_j(t_{\tilde{s}-1}), \tag{A14}$$

$$\tilde{R}_{ss}(t_s,t_{\tilde{s}})=\frac{1}{N}\sum_j i\hat{\phi}_j(t_{\tilde{s}})\sin\phi_j(t_{s-1})\sin\phi_j(t_{\tilde{s}-1}), \tag{A15}$$

$$\tilde{R}_{sc}(t_s,t_{\tilde{s}})=\frac{1}{N}\sum_j i\hat{\phi}_j(t_{\tilde{s}})\sin\phi_j(t_{s-1})\cos\phi_j(t_{\tilde{s}-1}), \tag{A16}$$

$$\tilde{R}_{cs}(t_s,t_{\tilde{s}})=\frac{1}{N}\sum_j i\hat{\phi}_j(t_{\tilde{s}})\cos\phi_j(t_{s-1})\sin\phi_j(t_{\tilde{s}-1}), \tag{A17}$$

$$\tilde{U}_{cc}(t_s,t_{\tilde{s}})=\frac{1}{N}\sum_j i\hat{\phi}_j(t_s)i\hat{\phi}_j(t_{\tilde{s}})\cos\phi_j(t_{s-1})\cos\phi_j(t_{\tilde{s}-1}), \tag{A18}$$

$$\tilde{U}_{ss}(t_s,t_{\tilde{s}})=\frac{1}{N}\sum_j i\hat{\phi}_j(t_s)i\hat{\phi}_j(t_{\tilde{s}})\sin\phi_j(t_{s-1})\sin\phi_j(t_{\tilde{s}-1}), \tag{A19}$$

$$\tilde{U}_{cs}(t_s,t_{\tilde{s}})=\frac{1}{N}\sum_j i\hat{\phi}_j(t_s)i\hat{\phi}_j(t_{\tilde{s}})\cos\phi_j(t_{s-1})\sin\phi_j(t_{\tilde{s}-1}), \tag{A20}$$

one can write the generating functional as

$$[Z]_{\omega,J}=\int \Pi_s\Pi_k\frac{d\phi_k(t_s)d\hat{\phi}_k(t_s)}{2\pi}\times\exp\Bigg\{-\tilde{S}_0+\frac{J^2}{2N}\sum_{s,\tilde{s}}\tau^2\tilde{U}_{cc}(t_s,t_{\tilde{s}})\tilde{K}_{ss}(t_s,t_{\tilde{s}})+\tilde{U}_{ss}(t_s,t_{\tilde{s}})\tilde{K}_{cc}(t_s,t_{\tilde{s}})-2\tilde{U}_{cs}(t_s,t_{\tilde{s}})\tilde{K}_{sc}(t_s,t_{\tilde{s}})-2\eta\tilde{R}_{cc}(t_s,t_{\tilde{s}})\tilde{R}_{ss}(t_s,t_{\tilde{s}})+2\eta\tilde{R}_{sc}(t_s,t_{\tilde{s}})\tilde{R}_{cs}(t_s,t_{\tilde{s}}). \tag{A21}$$

These terms can be written as squares using $4ab=(a+b)^2-(a-b)^2$:

$$[Z]_{\omega,J}=\int \Pi_s\Pi_k\frac{d\phi_k(t_s)d\hat{\phi}_k(t_s)}{2\pi}\exp\Bigg\{-\tilde{S}_0+\frac{J^2}{2N}\sum_{s,\tilde{s}}\tau^2\frac{1}{4}(\tilde{U}_{cc}(t_s,t_{\tilde{s}})+\tilde{R}_{ss}(t_s,t_{\tilde{s}}))^2-(\tilde{U}_{cc}(t_s,t_{\tilde{s}})-\tilde{R}_{ss}(t_s,t_{\tilde{s}}))^2+(\tilde{U}_{ss}(t_s,t_{\tilde{s}})+\tilde{K}_{cc}(t_s,t_{\tilde{s}}))^2-(\tilde{U}_{ss}(t_s,t_{\tilde{s}})-\tilde{K}_{cc}(t_s,t_{\tilde{s}}))^2-2(\tilde{U}_{cs}(t_s,t_{\tilde{s}})+\tilde{K}_{sc}(t_s,t_{\tilde{s}}))^2+2(\tilde{U}_{cs}(t_s,t_{\tilde{s}})-\tilde{K}_{sc}(t_s,t_{\tilde{s}}))^2-2\eta(\tilde{R}_{cc}(t_s,t_{\tilde{s}})+\tilde{R}_{ss}(t_s,t_{\tilde{s}}))^2+2\eta(\tilde{R}_{cc}(t_s,t_{\tilde{s}})-\tilde{R}_{ss}(t_s,t_{\tilde{s}}))^2+2\eta(\tilde{U}_{cs}(t_s,t_{\tilde{s}})+\tilde{K}_{sc}(t_s,t_{\tilde{s}}))^2-2\eta(\tilde{U}_{cs}(t_s,t_{\tilde{s}})-\tilde{K}_{sc}(t_s,t_{\tilde{s}}))^2\Bigg\}. \tag{A22}$$

Linearization of the exponents with a functional HST yields

$$[Z]_{\omega,J}=\int \Pi_s\Pi_k\frac{d\phi_k(t_s)d\hat{\phi}_k(t_s)}{2\pi}\int DQ_\alpha(t_s,t_{\tilde{s}})\times\exp\Bigg\{-\tilde{S}_0+\sum_{s,\tilde{s}}\tau^2\frac{-4N}{J^2}\sum_\alpha Q_\alpha^2(t_s,t_{\tilde{s}})+Q_1(t_s,t_{\tilde{s}})(\tilde{U}_{cc}(t_s,t_{\tilde{s}})+\tilde{R}_{ss}(t_s,t_{\tilde{s}}))\;\vdots\;-\sqrt{2\eta}Q_8(t_s,t_{\tilde{s}})(\tilde{U}_{cs}(t_s,t_{\tilde{s}})-\tilde{K}_{sc}(t_s,t_{\tilde{s}}))\Bigg\}. \tag{A23}$$

A linear coordinate transformation in Q_α is carried out

$$\left.\begin{aligned}\tilde{Q}_\alpha&:=Q_\alpha+iQ_{\alpha+1}\\ \tilde{Q}_{\alpha+1}&:=Q_\alpha-iQ_{\alpha+1}\end{aligned}\right\}\text{ for }\alpha=1,3,5,9,\qquad \left.\begin{aligned}\tilde{Q}_\alpha&:=iQ_\alpha+Q_{\alpha+1}\\ \tilde{Q}_{\alpha+1}&:=iQ_\alpha-Q_{\alpha+1}\end{aligned}\right\}\text{ for }\alpha=7. \tag{A24}$$

$[Z]_{\omega,J}$ can be written as

$$[Z]_{\omega,J}=\int \Pi_s\Pi_k\frac{d\phi_k(t_s)d\hat{\phi}_k(t_s)}{2\pi}\times\int D\tilde{Q}_\alpha(t_s,t_{\tilde{s}})e^{-S[\{\phi_j\},\{\hat{\phi}_j\},\tilde{Q}_\alpha]} \tag{A25}$$

with

$$S[\{\phi_j\},\{\hat{\phi}_j\},\tilde{Q}_\alpha]$$

$$= +\tilde{S}_0 - \Bigg\{ \sum_{s,\tilde{s}} \tau^2 \frac{-2N}{J^2} \sum_{\alpha\in\{1,3,5,9\}} \tilde{Q}_\alpha \tilde{Q}_{\alpha+1} - \tilde{Q}_7\tilde{Q}_8$$
$$+\tilde{Q}_1(t_s,t_{\tilde{s}})\tilde{U}_{cc}(t_s,t_{\tilde{s}}) + \tilde{Q}_2(t_s,t_{\tilde{s}})\tilde{K}_{ss}(t_s,t_{\tilde{s}})$$
$$+\tilde{Q}_3(t_s,t_{\tilde{s}})\tilde{U}_{ss}(t_s,t_{\tilde{s}}) + \tilde{Q}_4(t_s,t_{\tilde{s}})\tilde{K}_{cc}(t_s,t_{\tilde{s}})$$
$$+\sqrt{2}\tilde{Q}_5(t_s,t_{\tilde{s}})\tilde{U}_{cs}(t_s,t_{\tilde{s}}) + \sqrt{2}\tilde{Q}_6(t_s,t_{\tilde{s}})\tilde{K}_{sc}(t_s,t_{\tilde{s}})$$
$$+\sqrt{2\eta}\tilde{Q}_7(t_s,t_{\tilde{s}})\tilde{R}_{cc}(t_s,t_{\tilde{s}}) + \sqrt{2\eta}\tilde{Q}_8(t_s,t_{\tilde{s}})\tilde{R}_{ss}(t_s,t_{\tilde{s}})$$
$$+\sqrt{2\eta}\tilde{Q}_9(t_s,t_{\tilde{s}})\tilde{R}_{cs}(t_s,t_{\tilde{s}}) + \sqrt{2\eta}\tilde{Q}_{10}(t_s,t_{\tilde{s}})\tilde{R}_{sc}(t_s,t_{\tilde{s}})\Bigg\}. \tag{A26}$$

The integrals over $\tilde{Q}_\alpha$ are evaluated by the method of steepest descent. In the limit $N\to\infty$ one obtains the exact saddle point equations

$$\tilde{Q}_1^0(t_s,t_{\tilde{s}}) = \frac{J^2}{2}\langle \tilde{K}_{ss}(t_s,t_{\tilde{s}})\rangle, \tag{A27}$$

$$\tilde{Q}_2^0(t_s,t_{\tilde{s}}) = \frac{J^2}{2}\langle \tilde{U}_{cc}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_3^0(t_s,t_{\tilde{s}}) = \frac{J^2}{2}\langle \tilde{K}_{cc}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_4^0(t_s,t_{\tilde{s}}) = \frac{J^2}{2}\langle \tilde{U}_{ss}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_5^0(t_s,t_{\tilde{s}}) = -\sqrt{2}\frac{J^2}{2}\langle \tilde{K}_{sc}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_6^0(t_s,t_{\tilde{s}}) = -\sqrt{2}\frac{J^2}{2}\langle \tilde{U}_{cs}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_7^0(t_s,t_{\tilde{s}}) = -\sqrt{2\eta}\frac{J^2}{2}\langle \tilde{R}_{ss}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_8^0(t_s,t_{\tilde{s}}) = -\sqrt{2\eta}\frac{J^2}{2}\langle \tilde{R}_{cc}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_9^0(t_s,t_{\tilde{s}}) = \sqrt{2\eta}\frac{J^2}{2}\langle \tilde{R}_{sc}(t_s,t_{\tilde{s}})\rangle,$$

$$\tilde{Q}_{10}^0(t_s,t_{\tilde{s}}) = \sqrt{2\eta}\frac{J^2}{2}\langle \tilde{R}_{cs}(t_s,t_{\tilde{s}})\rangle.$$

The averages on the right-hand side of Eq. (A27) are defined as

$$\langle f(\phi,\hat{\phi})\rangle = \frac{\int \Pi_s\Pi_k \frac{d\phi_k(t_s)d\hat{\phi}_k(t_s)}{2\pi} f(\phi,\hat{\phi})e^{-S[\{\phi_j\},\{\hat{\phi}_j\},\tilde{Q}_\alpha]}}{\int \Pi_s\Pi_k \frac{d\phi_k(t_s)d\hat{\phi}_k(t_s)}{2\pi}, e^{-S[\{\phi_j\},\{\hat{\phi}_j\}\tilde{Q}_\alpha]}}. \tag{A28}$$

The exponent $S[\{\phi_j\},\{\hat{\phi}_j\},\tilde{Q}_\alpha]$ now contains the stationary values of $\tilde{Q}_\alpha$. It factorizes into a single sum of local quantities:

$$S[\{\phi_j\},\{\hat{\phi}_j\},\tilde{Q}_\alpha] = \sum_j S_1[\phi_j,\hat{\phi}_j,\tilde{Q}_\alpha] = [Z]_{\omega,J} \tag{A29}$$

can be written as

$$[Z]_{\omega,J} = \Pi_{i=1}^N [Z]_{\omega,J}^1 = ([Z]_{\omega,J}^1)^N. \tag{A30}$$

As $[Z]_{\omega,J}$ is a power of $[Z]_{\omega,J}^1$, it describes a system of N identical, noninteracting particles. Therefore the average $\langle\cdot\rangle$ can be considered as one-particle average. From now on we drop the site index:

$$[Z]_J^1 = \int \Pi_s \frac{d\phi(t_s)d\hat{\phi}(t_s)}{2\pi} e^{-S_1[\phi,\hat{\phi},K,R,U]}, \tag{A31}$$

with

$$
\begin{aligned}
-S_1[\phi,\hat{\phi},K,R,U]=&\frac{\sigma^2}{2}\sum_{s}\ \tau\hat{\phi}^2(t_s)+\sum_{s}\ \tau i\hat{\phi}(t_s)\frac{\phi(t_s)-\phi(t_{s-1})}{\tau}+\frac{\mu^2}{2}\sum_{s,\tilde{s}}\ \tau^2\ \hat{\phi}(t_s)\hat{\phi}(t_{\tilde{s}})\\
&+\sum_{s,\tilde{s}}\ \tau^2\frac{J^2}{2}i\hat{\phi}(t_s)i\hat{\phi}(t_{\tilde{s}})[\cos\ \phi(t_{s-1})\cos\ \phi(t_{\tilde{s}-1})K_{ss}(t_s,t_{\tilde{s}})\sin\phi(t_{s-1})\sin\phi(t_{\tilde{s}-1})K_{cc}(t_s,t_{\tilde{s}})\\
&-2\cos\ \phi(t_{s-1})\sin\phi(t_{\tilde{s}-1})K_{sc}(t_s,t_{\tilde{s}})]\\
&+\sum_{s}\ \tau\ \eta J^2 i\hat{\phi}(t_s)\sum_{\tilde{s}}\ \tau[-\cos\ \phi(t_{s-1})\cos\ \phi(t_{\tilde{s}-1})R_{ss}(t_s,t_{\tilde{s}})-\sin\phi(t_{s-1})\sin\phi(t_{\tilde{s}-1})R_{cc}(t_s,t_{\tilde{s}})\\
&+\cos\ \phi(t_{s-1})\sin\phi(t_{\tilde{s}-1})R_{sc}(t_s,t_{\tilde{s}})+\sin\phi(t_{s-1})\cos\ \phi(t_{\tilde{s}-1})R_{cs}(t_s,t_{\tilde{s}})]\\
&+\sum_{s,\tilde{s}}\ \tau^2\frac{J^2}{2}[\cos\ \phi(t_{s-1})\cos\ \phi(t_{\tilde{s}-1})U_{ss}(t_s,t_{\tilde{s}})+\sin\phi(t_{s-1})\sin\phi(t_{\tilde{s}-1})U_{cc}(t_s,t_{\tilde{s}})\\
&-2\cos\ \phi(t_{s-1})\sin\phi(t_{\tilde{s}-1})U_{sc}(t_s,t_{\tilde{s}})],
\end{aligned}
\tag{A32}
$$

where the correlation functions and response functions are

$$
K_{ss}(t_s,t_{\tilde{s}})=\langle\tilde{K}_{ss}(t_s,t_{\tilde{s}})\rangle=\langle\sin\phi(t_{s-1})\sin\phi(t_{\tilde{s}-1})\rangle \tag{A33}
$$

and analogical. The response functions obey $R(t_s,t_{\tilde{s}})=0$ for $t_s\leqslant t_{\tilde{s}}$. U_{ss}, U_{sc}, and U_{cc} vanish, because, for all t_s and $t_{\tilde{s}}$, either t_s or $t_{\tilde{s}}$ is larger than t_{s-1} *and* $t_{\tilde{s}-1}$. A nonvanishing U would violate causality. The one-dimensional averaged generating functional $[Z]^1_{J,\omega}$ corresponds to a dynamics, which obeys the generalized Langevin equation (13) with multidimensional Gaussian noise (14).

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